RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College affiliated to University of Calcutta)

M.A./M.Sc. SECOND SEMESTER EXAMINATION, MAY 2024

FIRST YEAR [BATCH 2023-25]

Date : 27/05/2024

MATHEMATICS

Time : 11.00 am – 1:00 pm

PAPER : CC 9

Full Marks : 50

Answer all questions, Maximum one can score is 50.

[All symbols bear their usual meanings unless otherwise mentioned. $(X, \mathcal{A}, \mu)$ is a complete measure space.]

1. Let $E \subseteq \mathbb{R}$ be a bounded set. Show that there is a G_δ set A containing E such that $m^*(A) = m^*(E)$. [5]

2. If a Lebesgue measurable set E has finite measure then show that for any $\varepsilon > 0$ E can be written as disjoint union of finite number of Lebesgue measurable sets each having mesure at most ε. [5]

3. If $E \subseteq \mathbb{R}$ has positive Lebesuge measure, show that there exists $A \subseteq E$ which is non-Lebesgue measurable. [5]

4. Let $E \subseteq X$ be a measurable set with $\mu(E) < +\infty$. Let $f : E \to \mathbb{R}^*$ be a μ - measurable function such that f is finite almost everywhere on E. Show that for each $\varepsilon > 0$, there exist $F \subseteq E, F \in \mathcal{A}$ with $\mu(E - F) < \varepsilon$ and a sequence $\{\psi_n\}$ of simple functions on E such that $\psi_n \to f$ uniformly on F. [8]

5. Let (X, a, μ) be a complete measure space and $\{f_n\}$ be an increasing sequence of non-negative extended real valued μ - measurable functions on X converging everywhere to a $\mu-$measurable function f. Show that $\lim\limits_{n\to\infty} \int\limits_X f_n \, d\mu = \int\limits_X f \, d\mu$.

   Is it always true that $\lim\limits_{n\to\infty} \int\limits_X (f - f_n) \, d\mu = 0$? Justify your answer. [7]

6. Let f be a measurable function on [0,1] $\subseteq \mathbb{R}$ such that $0 < f(x) < \infty$ for all $x \in [0,1]$. Show that

   $$\int_0^1 f \cdot \int_0^1 \frac{1}{f} \geq 1$$ [5]

7. Let $\{f_n\}$ be a sequence of $\mu-$measurable functions on $E \in \mathcal{A}$ such that $|f_n| \leq g$ almost everywhere on E $\forall n \in \mathbb{N}$, where g is a non-negative Lebesgue integrable function over E. Show that

   $$\int\limits_E \liminf f_n \leq \liminf \int\limits_E f_n \leq \limsup \int\limits_E f_n \leq \int\limits_E \limsup f_n.$$ [10]

8. State and prove Hölder's inequality for measurable functions over a complete measure space. [7]

9. Let $E \in \mathcal{A}$ with $\mu(E) < +\infty$ and $1 \leq p_1 < p_2 \leq \infty$. Show that if $\{f_n\} \to f$ in $L^{p_2}(E)$, then $\{f_n\} \to f$ in $L^{p_1}(E)$. [8]

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